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The semigroup of ultrafilters near an idempotent of a semitopological semigroup

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ABSTRACT

Let $(T,+)$ be a Hausdorff semitopological semigroup, S be a dense subsemigroup of T and e be an idempotent element of T. The set e_S^* of ultrafilters on S that converge to e is a semigroup under restriction of the usual operation $+$ on βT_d, the Stone–Čech compactification of the discrete semigroup T_d. We characterize the smallest ideal of $(e_S^*,+)$, and those sets "central" in $(e_S^*,+)$, that is, those sets which are members of minimal idempotents in $(e_S^*,+)$. We describe some combinatorial applications of those sets that are central in $(e_S^*,+)$.

1. Introduction

Given a discrete space $(T,+)$, we take the points of βT, the Stone–Čech compactification of T, to be the ultrafilters on T (or the ultrafilters in the collection of all subsets of T), with the points of T identified with the principal ultrafilters. The topology of βT can be defined by stating that the sets of the form $\{p \in \beta T\colon\ A \in p\}$, where A is a subset of T, are a base for the open sets. For any $p \in \beta T$ and any $A \subseteq T$, $A \in p$ if and only if $p \in \overline{A}$, where $\overline{A}$ denotes $cl_{\beta T} A$. If A is a subset of T, we shall use A^* to denote $\overline{A} - A$.

Let $p, q \in \beta T$ and $A \subseteq T$, then $A \in p+q$ if and only if $\{s \in T\colon\ -s+A \in q\} \in p$, where $-s+A = \{t \in T\colon\ s+t \in A\}$. For every $p \in \beta T$, the map $r_p : \beta T \to \beta T$ defined by $r_p(q) = q+p$ is continuous, for every $s \in T$, the map $\lambda_s : \beta T \to \beta T$ defined by $\lambda_s(q) = s+q$ is continuous. For more details see [11] and [2].

The ultrafilter semigroup $(0^+,+)$ of the topological semigroup $T = ((0,+\infty),+)$ consists of all nonprincipal ultrafilter on $T = (0,+\infty)$ converging to 0 and is a closed subsemigroup in the Stone–Čech compactification βT_d of T as a discrete semigroup. In [8], N. Hindman and I. Leader characterized the smallest ideal of $(0^+,+)$, its closure, and those sets "central" in $(0^+,+)$, that is, those sets which are members of minimal idempotents in $(0^+,+)$. They derived new combinatorial applications of those sets that are central in $(0^+,+)$. Related topics in [3] and [4] can be found.

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In Section 2 we define ultrafilters near a point and show that if $e \in T$ is an idempotent then

$$e^* = \left\{ p \in \beta T_d \colon e \in \bigcap_{A \in p} cl_T(A) \right\}$$

is a compact subsemigroup of βT_d. (Given a topological space X, the notation X_d represents the set X with the discrete topology.)

In Section 3 we characterize the members of the smallest ideal of $(e^*, +)$ and its closure. We also describe those subsets of T that have idempotents in $(e^*, +)$ in their closure.

Especially important in the combinatorial applications have been the central sets, i.e. those sets with idempotents in the intersection of their closure with the smallest ideal. In Section 4 we describe sets that are central near e, and we derive combinatorial results from their existence.

2. Preliminary

In this paper, $(T, +)$ denotes a Hausdorff semitopological semigroup that is not necessarily commutative. $E(T)$ denotes the collection of all idempotents in T. For every $x \in T$, τ_x denotes the collection of all neighborhoods of x, where a set U is called a neighborhood of $x \in T$ if $x \in int_T(U)$. For a set A, we write $\mathcal{P}_f(A)$ for the set of finite nonempty subsets of A and $\mathcal{P}(A)$ denote the collection of all subsets of A.

Definition 2.1. Let $(T, +)$ be a semitopological semigroup and S be a subsemigroup of T.

(a) Given $x \in T$, $x_S^* = \{p \in \beta S_d \colon x \in \bigcap_{A \in p} cl_T A\}$.
(b) $B(S) = \bigcup_{x \in T} x_S^*$.
(c) $\infty^* = \beta S_d \setminus B(S)$.

The set $B(S)$ is the set of "bounded" ultrafilters on S and the set ∞^* is the set of "unbounded" ultrafilters on S.

We say $p \in x^*$ is a near point to x. If x be a limit point of T, then $x^* \cap T^* \neq \emptyset$.

Lemma 2.2. *Let $(T, +)$ be a Hausdorff semitopological semigroup and S be a semigroup of T.*

(a) *Let $x \in T$. Then $x_S^* \neq \emptyset$ if and only if $x \in cl_T(S)$.*
(b) *$p \in \infty^*$ if and only if for each $x \in T$ there exists $A \in p$ such that $x \notin cl_T(A)$.*
(c) *If $\tau_x \subseteq p$ then $p \in x_S^*$.*
(d) *U is a neighborhood of x if and only if $U \in p$ for each $p \in x_S^*$.*
(e) *Let $A \subseteq T$. Then $x \in cl_T A$ if and only if $cl_{\beta T_d} A \cap x_S^* \neq \emptyset$.*

Proof. (a) It is obvious.

(b) Let $p \in \infty^*$, so $p \notin x^*$ for each $x \in T$. Hence $x \notin \bigcap_{A \in p} cl_T A$. Thus $x \notin cl_T A$ for some $A \in p$.

Conversely, suppose for each $x \in T$, there exists $A \in p$ such that $x \notin cl_T A$. Hence $p \notin B(T)$ and thus $p \in \infty^*$.

(c) Let $U \in p$ for each $U \in \tau_x$, thus $U \cap A \neq \emptyset$ for each $U \in \tau_x$ and for each $A \in p$. This implies $x \in cl_T A$ for each $A \in p$. Therefore $p \in x^*$.

(d) Let $U \in \tau_x$ and $p \in x^*$. $U \cap A \neq \emptyset$ for each $A \in p$, so $U \in p$.

Conversely, let $x \in \partial U = U - int_T(U)$, so $\{U^c \cup \partial U\} \cup \tau_x \subseteq p$, thus $p \in x^*$ and $U \notin p$, a contradiction.

(e) It is obvious. □

Lemma 2.3. *Let $(T, +)$ be a semitopological semigroup and S be a dense subsemigroup of T, then*:

(i) *For each $x, y \in T$, $x_S^* + y_S^* \subseteq (x + y)_S^*$.*
(ii) *Let $e \in T$ be an idempotent, then e_S^* is a compact subsemigroup of βS_d.*

Proof. (i) Pick $p \in x_S^*$ and pick $q \in y_S^*$. Suppose that $A \in p + q$, then $\{t \in S \colon -t + A \in q\} \in p$. Since $-t + A \in q$ so $y \in cl_T(-t + A)$. Therefore there exists a net $\{x_\alpha\} \subseteq -t + A$ such that $x_\alpha \to y$ and so $t + x_\alpha \to t + y$ for each $t \in \{s \in S \colon -s + A \in q\}$. Also since $x \in cl_T(\{t \in S \colon -t + A \in q\})$, so there exists a net $\{t_\beta\}$ such that $t_\beta \to x$. Thus $\lim_\beta \lim_\alpha t_\beta + x_\alpha = x + y$. Since for each α and β, $t_\beta + x_\alpha \in A$ so $x + y \in cl_T(A)$. This completes the proof.

(ii) It is obvious. □

3. Idempotents and the smallest ideal of e^*

Let S be a dense subsemigroup of a semitopological semigroup $(T, +)$, and $e \in E(T)$. Among the consequences of the fact that $(e_S^*, +)$ is a compact right topological semigroup is the fact that it contains idempotents [5, Corollary 2.10]. If S

is an arbitrary discrete semigroup, it is a result of Galvin's (see [6, Theorem 2.5]) that a set $A \subseteq S$ is a member of some idempotent in βS if and only if there is some sequence $\{x_n\}$ in S with $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$, where

$$FS\big(\langle x_n\rangle_{n=1}^{\infty}\big) = \left\{ \sum_{n\in F} x_n \colon\ F \text{ is a finite nonempty subset of } \mathbb{N} \right\}.$$

We have a similar characterization of members of idempotents in e_S^*.

Definition 3.1. Let $(T,+)$ be a semitopological semigroup.

(a) Let $\mathcal{B}$ be a local base at the point $x \in T$. We say $\mathcal{B}$ has the finite cover property if $\{V \in \mathcal{B}\colon\ y \in V\}$ is finite for each $y \in T - \{x\}$.
(b) Let S be a dense subsemigroup of T and $e \in E(T)$. Let $\{x_n\}$ be a sequence in S. We say $\sum_{n=1}^{\infty} x_n$ converges near e if for each $U \in \tau_e$ there exists $m \in \mathbb{N}$ such that $FS(\langle x_n\rangle_{n=k}^{l}) \subseteq U$ for each $l > k \geqslant m$.
(c) Let $\mathcal{B} = \{U_n\colon\ n \in \mathbb{N}\}$ be a countable local base at the point $x \in T$ such that $U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$, $U_{n+1} + U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$, and for each sequence $\{x_n\}_{n=1}^{\infty}$ if $x_n \in U_n$ for each $n \in \mathbb{N}$ then $\sum_{n=1}^{\infty} x_n$ converges near x. Then we say $\mathcal{B}$ is a countable local base for convergence at the point $x \in T$.
(d) Let $\mathcal{B}$ be a local base at the point $x \in T$. If $\mathcal{B}$ satisfies in conditions (a) and (c) then $\mathcal{B}$ is called a countable local base that has the finite cover property for convergence at the point x. For simplicity we say $\mathcal{B}$ has the **F** property at the point x.

Theorem 3.2. *Let $(T,+)$ be a semitopological semigroup and S be a dense subsemigroup of T, $e \in E(T)$ and $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ have the* **F** *property at the point e. Then there exists $p = p + p$ in e_S^* with $A \in p$ if and only if there is some sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that $\lim_{n\to\infty} x_n = e$, $\sum_{n=1}^{\infty} x_n$ converges near e and $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Necessity. Let $A_1 = A$ and $B_1 = \{x \in S\colon\ -x + A_1 \in p\}$ so $B_1 \in p$. Since $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ is a countable local base for convergence at the point e, pick $x_1 \in B_1 \cap A_1 \cap U_1$ and let $A_2 = A_1 \cap (-x_1 + A_1)$. Inductively, given $A_n \in p$, let $B_n = \{x \in S\colon\ -x + A_n \in p\}$ and pick $x_n \in B_n \cap A_n \cap U_n$. Let $A_{n+1} = A_n \cap (-x_n + A_n)$. Then one easily sees that $\langle x_n\rangle_{n=1}^{\infty}$ is as required.

Sufficiency. Without losing of generality, suppose that $x_n \in U_n$, $\lim_{n\to\infty} x_n = e$ and $J = \bigcap_{m=1}^{\infty} cl_{\beta S_d} FS(\langle x_n\rangle_{n=m}^{\infty})$. By Lemma 5.11 in [11], J is a subsemigroup of βS_d and contains an idempotent p. Since $\sum_{n=1}^{\infty} x_n$ converges, for each $U \in \tau_e$, there are m and k in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=m}^{\infty}) \subseteq U \cap S$ and $U_k \subseteq U$. Then for each $r_1 > \cdots > r_k > k+1$ implies that

$$x_{r_1} + x_{r_2} \in U_{r_1} + U_{r_2} \subseteq U_{r_2 - 1}$$

and,

$$(x_{r_1} + x_{r_2}) + x_{r_3} \in U_{r_2 - 1} + U_{r_3} \subseteq U_{r_3} + U_{r_3} \subseteq U_{r_3 - 1}.$$

Hence $x_{r_1} + \cdots + x_{r_k} \in U_k \subseteq U$. Thus for each $k \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $FS(\langle x_n\rangle_{n=m}^{\infty}) \subseteq U_k \cap S$. Consequently $J \subseteq e_S^*$. $\square$

As a compact right topological semigroup, e_S^* has a smallest two sided ideal [1, Theorem 1.3.11]. We turn our attention to characterizing the smallest ideal of e_S^* and its closure. Deducing the parallels with known theory becomes progressively less straightforward as we proceed. If $(S,+)$ is a discrete semigroup we know from [7, Corollary 3.6] that any $p \in \beta S$ is in the smallest ideal of βS if and only if, for each $A \in p$, $\{x \in S\colon\ -x + A \in p\}$ is syndetic. (A subset B of S is syndetic if and only if there is a finite nonempty subset F of S such that $S \subseteq \bigcup_{t\in F} -t + B$. The terminology comes from topological dynamics.)

Definition 3.3. Let S be a dense subsemigroup of $(T,+)$ and $e \in E(T)$.

(a) K is the smallest ideal of e_S^*.
(b) A subset B of S is syndetic near e if and only if for each $U \in \tau_e$, there exist some $F \in \mathcal{P}_f(U \cap S)$ and some $V \in \tau_e$ such that $S \cap V \subseteq \bigcup_{t\in F}(-t + B)$.

Theorem 3.4. *Let S be a dense subsemigroup of T and let $p \in e_S^*$, where $e \in E(T)$. The following statements are equivalent.*

(a) *$p \in K$.*
(b) *For all $A \in p$, $\{x \in S\colon\ -x + A \in p\}$ is syndetic near e.*
(c) *For all $r \in e_S^*$, $p \in e_S^* + r + p$.*

Proof. (a) implies (b). Let $A \in p$, $B = \{x \in S\colon -x + A \in p\}$ and suppose that B is not syndetic near e. Pick $U \in \tau_e$ such that for each $F \in \mathcal{P}_f(U \cap S)$ and each $V \in \tau_e$, $(S \cap V) \setminus \bigcup_{t \in F}(-t + B) \neq \emptyset$. Let

$$\mathcal{G} = \left\{ S \cap V \setminus \bigcup_{t \in F}(-t + B)\colon F \in \mathcal{P}_f(U \cap S) \text{ and } V \in \tau_e \right\}.$$

Then $\mathcal{G}$ has the finite intersection property, so pick $q \in \beta S_d$ with $\mathcal{G} \subseteq q$. Since $\{S \cap V\colon V \in \tau_e\} \subseteq q$, thus $q \in e_S^*$, by Lemma 2.2.

Pick a minimal left ideal L of e_S^* with $L \subseteq e_S^* + q$, by [1, Proposition 1.2.4]. Since K is the union of all of the minimal right ideal of e_S^* [1, Theorem 1.3.11], pick a minimal right ideal R of e_S^* with $p \in R$. Then $L \cap R$ is a group [1, Theorem 1.3.11], so let r be the identity of $L \cap R$. Therefore $p \in r + e_S^* = R$ and $p = r + p$ so $B \in r$. Also $r \in e_S^* + q$ so pick $w \in e_S^*$ such that $r = w + q$. Then $U \cap S \in w$ and $\{t \in S\colon -t + B \in q\} \in w$ so pick $t \in U \cap S$ such that $-t + B \in q$. But $(S \cap U) \setminus (-t + B) \in \mathcal{G} \subseteq q$ is a contradiction.

(b) implies (c). Let $A \in p$ and let $B = \{x \in S\colon -x + A \in p\}$. Let $r \in e_S^*$. Assume that, for every $U \in \tau_e$, there exists $F_U \in \mathcal{P}_f(U \cap S)$ and $V_U \in \tau_e$ such that $S \cap V_U \subseteq \bigcup_{t \in F_U}(-t + B)$. Since $S \cap V_U \in r$, there exists $t_U \in F_U$ such that $t_U + r + p \in \overline{A}$. If x is a limit point in βS_d of the net $\{t_U\}_{U \in \tau_e}$, then $x \in e_S^*$ and $x + r + p = p$.

(c) implies (a). Pick $r \in K$. □

Definition 3.5. Let S be a dense subsemigroup of $(T, +)$ and $e \in E(T)$. A subset B of S is topologically piecewise syndetic near e if and only if for each $U \in \tau_e$ there exist some $F \in \mathcal{P}_f(U \cap S)$ and some $V \in \tau_e$ such that for each $G \in \mathcal{P}_f(S)$ and $O \in \tau_e$ there exists $x \in S \cap O$ such that $(G \cap V) + x \subseteq \bigcup_{t \in F}(-t + B)$.

Remark. A subset B of S is topologically piecewise syndetic near e if and only if, for every $U \in \tau_e$, there exist $G \in \mathcal{P}_f(U \cap S)$ and $V \in \tau_e$ such that

$$\left\{ -a + \left(\bigcup_{t \in G} -t + B \right)\colon a \in V \cap S \right\} \cup \tau_e$$

has the finite intersection property.

Let $B \subseteq S$ that for every $U \in \tau_e$, there exist $G \in \mathcal{P}_f(U \cap S)$ and $V \in \tau_e$ have the property that

$$\left\{ -a + \left(\bigcup_{t \in G} -t + B \right)\colon a \in V \cap S \right\} \cup \tau_e$$

has the finite intersection property. Therefore for each $F \in \mathcal{P}_f(V \cap S)$ and $O \in \tau_e$,

$$\left(\bigcap_{a \in F} \left(-a + \left(\bigcup_{t \in G} -t + B \right) \right) \right) \cap O \neq \emptyset.$$

Therefore there exists $x \in O \cap S$ such that

$$x \in \bigcap_{a \in F} \left(-a + \left(\bigcup_{t \in G} -t + B \right) \right)$$

and so $F + x \subseteq \bigcup_{t \in G} -t + B$. Thus for each $O \in \tau_e$ and $F \in \mathcal{P}_f(S)$ there exists $x \in O \cap S$ such that

$$(F \cap V) + x \subseteq \bigcup_{t \in G}(-t + B).$$

So B be topologically piecewise syndetic near e.

Now let B is topologically piecewise syndetic near e. So for each $U \in \tau_e$ there exist $F \in \mathcal{P}_f(U \cap S)$ and $V \in \tau_e$ such that for each $G \in \mathcal{P}_f(S)$ and $O \in \tau_e$ there exists $x \in O \cap S$ that

$$(G \cap V) + x \subseteq \bigcup_{t \in F}(-t + B).$$

Thus $x \in \bigcap_{y \in G \cap V}(-y + (\bigcup_{t \in F}(-t + B)))$, and so

$$\bigcap_{y \in G \cap V} \left(-y + \left(\bigcup_{t \in F}(-t + B) \right) \right) \cap O \neq \emptyset.$$

Therefore

$$\left\{-a+\left(\bigcup_{t\in G}-t+B\right):\ a\in V\cap S\right\}\cup\tau_e$$

has the finite intersection property.

Theorem 3.6. *Let S be a dense subsemigroup of $(T,+)$, $e\in E(T)$, and $A\subseteq S$. Then $K\cap cl_{\beta S}(A)\neq\emptyset$ if and only if A is topologically piecewise syndetic near e.*

Proof. Necessity. Pick $p\in K\cap cl_{\beta S}A$. Let $B=\{x\in S:\ -x+A\in p\}$. By Theorem 3.4, B is syndetic near e. Then for each $U\in\tau_e$ there exist some $F\in\mathcal{P}_f(U\cap S)$ and $V\in\tau_e$ such that $(S\cap V)\subseteq\bigcup_{t\in F}(-t+B)$. Let $G\in\mathcal{P}_f(S)$. If $G\cap V=\emptyset$, the conclusion is trivial, so assume $G\cap V\neq\emptyset$ and let $H=G\cap V$. For each $y\in H$, we have $y\in\bigcup_{t\in F}(-t+B)$ so pick $t_y\in F$ such that $y\in -t_y+B$. Then $-(t_y+y)+A\in p$. By Lemma 2.2, for $O\in\tau_e\subseteq p$, pick $x\in O\cap\bigcap_{y\in H}(-(t_y+y)+A)$. Then $t_y+y+x\in A$ so $y+x\in -t_y+A\subseteq\bigcup_{t\in F}(-t+B)$. Thus for each $U\in\tau_e$ there exist some $F\in\mathcal{P}_f(U\cap S)$ and some $V\in\tau_e$ such that for all $G\in\mathcal{P}_f(S)$ and $O\in\tau_e$ there exists $x\in S\cap O$ such that $(G\cap V)+x\subseteq\bigcup_{t\in F}(-t+B)$.

Sufficiency. For each $U\in\tau_e$ there exist some $F_U\in\mathcal{P}_f(U\cap S)$ and some $V_U\in\tau_e$ such that for each $G\in\mathcal{P}_f(S)$ and $W\in\tau_e$ there exists $x\in S\cap W$ such that $(G\cap V_U)+x\subseteq\bigcup_{t\in F_U}(-t+B)$.

For each $G\in\mathcal{P}_f(S)$ and $U\in\tau_e$, let

$$C(G,U)=\left\{x\in U\cap S:\ (G\cap V_U)+x\subseteq\bigcup_{t\in F_U}(-t+A)\right\}.$$

By assumption $C(G,U)\neq\emptyset$. It is obvious that for $G_1,G_2\in\mathcal{P}_f(S)$ and $U_1,U_2\in\tau_e$, we will have $C(G_1\cup G_2,U_1\cap U_2)\subseteq C(G_1,U_1)\cap C(G_2,U_2)$. Thus $\mathcal{C}=\{C(G,U):\ G\in\mathcal{P}_f(S)\text{ and }U\in\tau_e\}$ has the finite intersection property, hence pick $p\in\beta S_d$ such that $\mathcal{C}\subseteq p$. So $p\in e_S^*$ (by Lemma 2.2). Now we claim that for each $U\in\tau_e$,

$$e_S^*+p\subseteq cl_{\beta S_d}\left(\bigcup_{t\in F_U}(-t+A)\right).$$

Let $q\in e_S^*$. To show that $\bigcup_{t\in F_U}(-t+A)\in q+p$, we show that

$$V_U\cap S\subseteq\left\{y\in S:\ -y+\bigcup_{t\in F_U}(-t+A)\in p\right\}.$$

Let $y\in V_U\cap S$, so $C(\{y\},U)\in p$ and $C(\{y\},U)\subseteq -y+\bigcup_{t\in F_U}(-t+A)$. Now pick $r\in(e_S^*+p)\cap K$ (since e_S^*+p is a left ideal of e_S^*). Given $t_U\in F_U\subseteq U$ such that $-t_U+A\in r$, pick $q\in e_S^*\cap cl_{\beta S_d}\{t_U:\ U\in\tau_e\}$. Then $q+r\in K$ and $\{t_U:\ U\in\tau_e\}\subseteq\{t\in S:\ -t+A\in r\}$. So $A\in q+r$ as required. □

Definition 3.7. Let S be a dense subsemigroup of $(T,+)$, $e\in E(T)$ and $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ have the **F** property at the point e. A subset A of S is a piecewise syndetic near e with countable base if and only if there exists $\{F_n\}_{n\in\mathbb{N}}$ such that

(1) for each $n\in\mathbb{N}$, $F_n\in\mathcal{P}_f(U_n\cap S)$, and
(2) for each $G\in\mathcal{P}_f(S)$ and each $W\in\tau_e$ there is some $x\in W\cap S$ such that for all $n\in\mathbb{N}$, $(G\cap U_n)+x\subseteq\bigcup_{t\in F_n}(-t+A)$.

Theorem 3.8. *Let S be a dense subsemigroup of $(T,+)$, $e\in E(T)$, and $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ have the* **F** *property at the point e. Let $A\subseteq S$. Then $K\cap cl_{\beta S}A\neq\emptyset$ if and only if A is a piecewise syndetic near e with countable base.*

Proof. Necessity. Pick $p\in K\cap cl_{\beta S}A$. Let $B=\{x\in S:\ -x+A\in p\}$. By Theorem 3.4, B is syndetic near e. Since $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ has the **F** property at the point e, inductively for $n\in\mathbb{N}$ pick $F_n\in\mathcal{P}_f(U_n\cap S)$ and $V_n\subseteq U_n$ such that $S\cap V_n\subseteq\bigcup_{t\in F_n}(-t+B)$. Let $G\in\mathcal{P}_f(S)$ be given. If $G\cap V_1=\emptyset$, the conclusion is trivial, so assume $G\cap V_1\neq\emptyset$ and assume that $H=G\cap V_1$. For each $y\in H$, let $m(y)=\sup\{n:\ y\in V_n\}$. For each $y\in H$ and each $n\in\{1,\dots,m(y)\}$, we have $y\in\bigcup_{t\in F_n}(-t+B)$ so pick $t(y,n)\in F_n$ such that $y\in -t(y,n)+B$. Then given $y\in H$ and $n\in\{1,\dots,m(y)\}$, one has $-(t(y,n)+y)+A\in p$. Now let $W\in\tau_e$ be given. Then $W\cap S\in p$ so pick

$$x\in W\cap\bigcap_{y\in H}\bigcap_{n=1}^{m(y)}\bigl(-\bigl(t(y,n)+y\bigr)+A\bigr).$$

Then given $n\in\mathbb{N}$ and $y\in G\cap U_n$, one has $y\in H$ and $n\leqslant m(y)$ so $t(y,n)+y+x\in A$ so

$$y+x\in -t(y,n)+A\subseteq\bigcap_{t\in F_n}(-t+A).$$

Sufficiency. Pick $\{F_n\}$ satisfying (1) and (2) of Definition 3.7. For each $G \in \mathcal{P}_f(S)$ and $W \in \tau_e$, let

$$C(G,W) = \left\{x \in W \cap S\colon \text{ for all } n \in \mathbb{N},\ (G \cap U_n) + x \subseteq \bigcup_{t \in F_n} (-t + A)\right\}.$$

By assumption each $C(G,W) \neq \emptyset$. Further, given $G_1, G_2 \in \mathcal{P}_f(S)$ and $W_1, W_2 \in \tau_e$, one has $C(G_1 \cup G_2, W_1 \cap W_2) \subseteq C(G_1, W_1) \cap C(G_2, W_2)$ so $\{C(G,W)\colon G \in \mathcal{P}_f(S) \text{ and } W \in \tau_e\}$ has the finite intersection property so pick $p \in \beta S_d$ with $\{C(G,W)\colon G \in \mathcal{P}_f(S) \text{ and } W \in \tau_e\} \subseteq p$. Note that since each $C(G,W) \subseteq W \cap S$, one has $p \in e_S^*$. Now we claim that for each $n \in \mathbb{N}$, $e_S^* + p \subseteq cl_{\beta S_d}(\bigcup_{t \in F_n}(-t+A))$, so let $n \in \mathbb{N}$ and let $q \in e_S^*$. To show that $\bigcup_{t \in F_n}(-t+A) \in q + p$, we show that

$$U_n \cap S \subseteq \left\{y \in S\colon -y + \bigcup_{t \in F_n}(-t+A) \in p\right\}.$$

So let $y \in U_n \cap S$. Then $C(\{y\}, U_n) \in p$ and

$$C\big(\{y\}, U_n\big) \subseteq -y + \bigcup_{t \in F_n}(-t+A).$$

Now pick $r \in (e_S^* + p) \cap K$ (since $e_S^* + p$ is a left ideal of e_S^*). Given $n \in \mathbb{N}$, $\bigcup_{t \in F_n}(-t+A) \in r$ so pick $t_n \in F_n$ such that $-t_n + A \in r$. Now each $t_n \in F_n \subseteq U_n$ so $\lim_{n\to\infty} t_n = e$ hence pick $q \in e_S^* \cap cl_{\beta S_d}\{t_n\colon n \in \mathbb{N}\}$. Then $q + r \in K$ and $\{t_n\colon n \in \mathbb{N}\} \subseteq \{t \in S\colon -t + A \in r\}$ thus $A \in q + r$. □

We characterize those sets that are members of minimal idempotents in $(e_S^*, +)$. By [9, Theorem 2.1], there exists a relatively complicated characterization of those subsets $\mathcal{A}$ of $\mathcal{P}(S)$ which extend to a member of the smallest ideal of βS. This characterization was called “collection wise piecewise syndetic” in [10] and used there to characterize central sets, that is members of idempotents in the smallest ideal of βS.

Definition 3.9. Let S be a dense subsemigroup of $(T,+)$ and $e \in E(T)$. Let $\tau_e(S) = \{U \cap S\colon U \in \tau_e\}$.

(a) A set $A \subseteq S$ is central near e if and only if there is some idempotent $p \in K$ with $A \in p$.
(b) A family $\mathcal{A} \subseteq \mathcal{P}(S)$ is topological collectionwise piecewise syndetic near e if and only if there exist functions

$$F : \mathcal{P}_f(\mathcal{A}) \to \prod_{U \in \tau_e} \mathcal{P}_f(U \cap S)$$

and

$$\delta : \mathcal{P}_f(\mathcal{A}) \to \prod_{U \in \tau_e} \big(\mathcal{P}(U) \cap \tau_e\big)$$

such that for every $O \in \tau_e$, every $G \in \mathcal{P}_f(S)$, and every $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$, there is some $t \in S \cap O$ such that for every $U \in \tau_e$ and every $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$,

$$\big(G \cap \delta(\mathcal{F})_U\big) + t \subseteq \bigcup_{x \in F(\mathcal{F})_U} \Big(-x + \bigcap \mathcal{F}\Big).$$

(c) Let $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ has the **F** property at the point e. A family $\mathcal{A} \subseteq \mathcal{P}(S)$ is collectionwise syndetic near e with countable base if and only if there exist functions

$$F : \mathcal{P}_f(\mathcal{A}) \to \prod_{n=1}^{\infty} \mathcal{P}_f(U_n \cap S)$$

and

$$\delta : \mathcal{P}_f(\mathcal{A}) \to \prod_{n=1}^{\infty} \big(\mathcal{P}(U_n) \cap \tau_e(S)\big)$$

such that for every $U \in \tau_e$, every $G \in \mathcal{P}_f(S)$, and every $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$, there is some $t \in S \cap U$ such that for every $n \in \mathbb{N}$ and every $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$,

$$\big(G \cap \delta(\mathcal{F})_n\big) + t \subseteq \bigcup_{x \in F(\mathcal{F})_n} \big(-x + \bigcap \mathcal{F}\big).$$

Theorem 3.10. *Let S be a dense subsemigroup of $(T,+)$, $e\in E(T)$, and $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ have the* **F** *property at the point e. There exists $p\in K$ such that $\mathcal{A}\subseteq p$ if and only if $\mathcal{A}$ is collectionwise piecewise syndetic near e with countable base.*

Proof. Necessity. Let $B(\mathcal{F})=\{x\in S\colon\ -x+\bigcap\mathcal{F}\in p\}$, for each $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$. Then by Theorem 3.4, $B(\mathcal{F})$ is syndetic near e with countable base, so for each $n\in\mathbb{N}$, pick $F(\mathcal{F})_n\in\mathcal{P}_f(U_n\cap S)$ and $\delta(\mathcal{F})_n\in\mathcal{P}(U_n)\cap\tau_e(S)$ such that

$$S\cap\delta(\mathcal{F})_n\subseteq\bigcup_{x\in F(\mathcal{F})_n}\bigl(-x+B(\mathcal{F})\bigr).$$

We have thus defined

$$F:\mathcal{P}_f(\mathcal{A})\to\prod_{n=1}^{\infty}\mathcal{P}_f(U_n\cap S)$$

and

$$\delta:\mathcal{P}_f(\mathcal{A})\to\prod_{n=1}^{\infty}\bigl(\mathcal{P}(U_n)\cap\tau_e(S)\bigr).$$

To see that these functions are as required, let $U\in\tau_e$, $G\in\mathcal{P}_f(S)$, and $\mathcal{H}\in\mathcal{P}_f(\mathcal{A})$ be given.

Pick $m\in\mathbb{N}$ such that $\max\{n\in\mathbb{N}\colon\ U_n\cap G\neq\emptyset\}<m$. For each $(y,n,\mathcal{F})$ such that $n\in\{1,2,\ldots,m\}$, $\mathcal{F}\in\mathcal{P}_f(\mathcal{H})$, and $y\in\delta(\mathcal{F})_n\cap G$, pick $x(y,n,\mathcal{F})\in F(\mathcal{F})_n$ such that $x(y,n,\mathcal{F})+y\in B(\mathcal{F})$, that is $-(x(y,n,\mathcal{F})+y)+\bigcap\mathcal{F}\in p$. Let

$$\mathcal{C}=\Bigl\{-\bigl(x(y,n,\mathcal{F})+y\bigr)+\bigcap\mathcal{F}\colon\ n\in\{1,2,\ldots,m\},\ \mathcal{F}\in\mathcal{P}_f(\mathcal{H}),\ \text{and}\ y\in\delta(\mathcal{F})_n\cap G\Bigr\}.$$

If $\mathcal{C}=\emptyset$, the conclusion is trivial, so we may assume $\mathcal{C}\neq\emptyset$ and hence $\mathcal{C}\in\mathcal{P}_f(p)$. Pick $t\in\bigcap\mathcal{C}\cap U$. Let $n\in\mathbb{N}$ and $\mathcal{F}\in\mathcal{P}_f(\mathcal{H})$ be given. If $G\cap\delta(\mathcal{F})_n=\emptyset$ the conclusion holds, so assume $G\cap\delta(\mathcal{F})_n\neq\emptyset$ and let $y\in G\cap\delta(\mathcal{F})_n$. Then $y\in\delta(\mathcal{F})_n$ and $y\in G$ so $n<m$. Thus $t\in-(x(y,n,\mathcal{F})+y)+\bigcap\mathcal{F}$ so

$$y+t\in-x(y,n,\mathcal{F})+\bigcap\mathcal{F}\subseteq\bigcup_{x\in F(\mathcal{F})_n}\Bigl(-x+\bigcap\mathcal{F}\Bigr).$$

Sufficiency. Pick functions F and δ as guaranteed by the assumption that $\mathcal{A}$ is collectionwise piecewise syndetic near e with countable base. Given $U\in\tau_e$, $G\in\mathcal{P}_f(S)$, and $\mathcal{H}\in\mathcal{P}_f(\mathcal{A})$, pick $t(\mathcal{H},G,U)\in U\cap S$ such that for every $n\in\mathbb{N}$ and $\mathcal{F}\in\mathcal{P}_f(\mathcal{H})$,

$$\bigl(G\cap\delta(\mathcal{F})_n\bigr)+t(\mathcal{H},G,U)\subseteq\bigcup_{x\in F(\mathcal{F})_n}\Bigl(-x+\bigcap\mathcal{F}\Bigr).$$

For each $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$ and every $y\in S$, let

$$D(\mathcal{F},y)=\bigl\{t(\mathcal{H},G,U)\colon\ \mathcal{H}\in\mathcal{P}_f(\mathcal{A}),\ G\in\mathcal{P}_f(S),\ y\in G,\ \mathcal{F}\subseteq\mathcal{H},\ \text{and}\ U\in\tau_e\bigr\}.$$

Then $D=\{D(\mathcal{F},y)\colon\ \mathcal{F}\in\mathcal{P}_f(\mathcal{A})\ \text{and}\ y\in S\}\cup\{U\cap S\colon\ U\in\tau_e\}$ has the finite intersection property. Indeed, given $\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n$, $y_1,y_2,\ldots,y_n$ and $U_1,U_2,\ldots,U_n\in\tau_e$, let

$$\mathcal{H}=\bigcup_{i=1}^{n}\mathcal{F}_i,\qquad G=\{y_1,y_2,\ldots,y_n\},\quad\text{and}\quad U=\bigcap_{k=1}^{n}U_k.$$

Then $t(\mathcal{H},G,U)\in\bigcap_{i=1}^{n}(D(\mathcal{F}_i,y_i)\cap U_i)$. So pick $u\in e_S^*$ such that $\{D(\mathcal{F},y)\colon\ \mathcal{F}\in\mathcal{P}_f(\mathcal{A})\ \text{and}\ y\in S\}\subseteq u$. Now we claim that for each $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$ and each $n\in\mathbb{N}$,

$$e_S^*+u\subseteq\bigcup_{x\in F(\mathcal{F})_n}cl_{\beta S_d}\Bigl(-x+\bigcap\mathcal{F}\Bigr).$$

So pick $q\in e_S^*$ and let $A=\bigcup_{x\in F(\mathcal{F})_n}(-x+\bigcap\mathcal{F})$. We claim that $\delta(\mathcal{F})_n\subseteq\{y\in S\colon\ -y+A\in u\}$, so that since $\delta(\mathcal{F})_n\in q$, we have $A\in q+u$. Let $y\in\delta(\mathcal{F})_n$. It suffices to show that $D(\mathcal{F},y)\subseteq-y+A$. So let $\mathcal{H}\in\mathcal{P}_f(\mathcal{A})$ with $\mathcal{F}\subseteq\mathcal{H}$, let $G\in\mathcal{P}_f(\mathcal{A})$ with $y\in G$, and let $U\in\tau_e$ be given. Then $y\in G\cap\delta(\mathcal{F})_n$ so $y+t(\mathcal{H},G,U)\in A$ as required. Pick a minimal left ideal L of e_S^* with $L\subseteq e_S^*+q$. Then

$$L\subseteq\bigcap_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}\bigcap_{n=1}^{\infty}\bigcup_{x\in F(\mathcal{F})_n}cl_{\beta S_d}\Bigl(-x+\bigcap\mathcal{F}\Bigr).$$

Pick $r \in L$. For each $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$ and each $n \in \mathbb{N}$, pick $x(\mathcal{F}, n) \in F(\mathcal{F})_n$ such that $-x(\mathcal{F}, n) + \bigcap \mathcal{F} \in r$. For each $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, let

$$\mathcal{E}(\mathcal{F}) = \{x(\mathcal{H}, n)\colon \mathcal{H} \in \mathcal{P}_f(\mathcal{A}),\ \mathcal{F} \subseteq \mathcal{H},\ \text{and}\ n \in \mathbb{N}\}.$$

We claim that $\{\mathcal{E}(\mathcal{F})\colon \mathcal{F} \in \mathcal{P}_f(\mathcal{A})\} \cup \tau_e(S)$ has the finite intersection property. Indeed, given $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ and $V_1, V_2, \ldots, V_n \in \tau_e(S)$, pick $n \in \mathbb{N}$ such that $U_n \subseteq \bigcap_{k=1}^m V_k$ and let $\mathcal{H} = \bigcup_{i=1}^m \mathcal{F}_i$. Then we have $x(\mathcal{H}, n) \in \bigcap_{i=1}^m \mathcal{E}(\mathcal{F}_i) \cap \bigcap_{i=1}^m V_i$. So pick $w \in e_S^*$ such that $\{\mathcal{E}(\mathcal{F})\colon \mathcal{F} \in \mathcal{P}_f(\mathcal{A})\} \subseteq w$. Let $p = w + r$. Then $p \in L \subseteq K$. To see that $\mathcal{A} \subseteq p$, let $A \in \mathcal{A}$. We show that $\mathcal{E}(\{A\}) \subseteq \{x \in S\colon -x + A \in r\}$. Let $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$ with $A \in \mathcal{F}$ and let $n \in \mathbb{N}$. Then $-x(\mathcal{F}, n) + \bigcap \mathcal{F} \in r$ and therefore $-x(\mathcal{F}, n) + \bigcap \mathcal{F} \subseteq -x(\mathcal{F}, n) + A$. □

Conjecture. *Let S be a dense subsemigroup of $(T, +)$ and $e \in E(T)$. Let $\mathcal{A} \subseteq \mathcal{P}(S)$. Then there exists $p \in K$ such that $\mathcal{A} \subseteq p$ if and only if $\mathcal{A}$ is topological collectionwise piecewise syndetic near e.*

We formalize the notion of "tree" below. We write $\omega = \{0, 1, 2, 3, \ldots\}$, the first infinite ordinal and recall that each ordinal is the set of its predecessors. (So $3 = \{0, 1, 2\}$ and $0 = \emptyset$ and, if f is the function

$$\{(0, 3), (1, 5), (2, 9), (3, 7), (4, 5)\}$$

then $f|_3 = \{(0, 3), (1, 5), (2, 9)\}$.)

Definition 3.11. *Tr* is a tree in A if and only if *Tr* is a set of functions and for each $f \in Tr$, $\mathrm{domain}(f) \in \omega$ and $\mathrm{range}(f) \subset A$ and if $\mathrm{domain}(f) = n > 0$, then $f|_{n-1} \in Tr$. *Tr* is a tree if and only if for some A, *Tr* is a tree in A.

Definition 3.12.

(a) Let f be a function with $\mathrm{domain}(f) = n \in \omega$ and let x be given. Then $f^\frown x = f \cup \{(n, x)\}$.
(b) Given a tree *Tr* and $f \in Tr$, $B_f = B_f(Tr) = \{x\colon\ f^\frown x \in Tr\}$.
(c) Let $(S, +)$ be a semigroup and let $A \subseteq S$. Then *Tr* is a $*$-tree in A if and only if *Tr* is a tree in A and all $f \in Tr$ and all $x \in B_f$, $B_{f^\frown x} \subseteq -x + B_f$.
(d) Let $(S, +)$ be a semigroup and let $A \subseteq S$. Then *Tr* is an *FS*-tree in A if and only if *Tr* is a tree in A and for all $f \in Tr$,

$$B_f = \left\{\sum_{t \in F} g(t)\colon\ g \in Tr,\ f \subsetneq g,\ \text{and}\ \emptyset \neq F \subseteq \mathrm{domain}(g) \setminus \mathrm{domain}(f)\right\}.$$

Lemma 3.13. *Let $(S, +)$ be a discrete semigroup and let p be an idempotent in βS_d. If $A \in p$, then there is an FS-tree Tr in A such that for each $f \in Tr$, $B_f \in p$.*

Proof. This is [10, Lemma 3.6]. □

Lemma 3.14. *Any FS-tree is a $*$-tree.*

Proof. This is [8, Lemma 4.6]. □

When we say that $\langle C_F \rangle_{F \in I}$ is a "downward directed family" we mean that I is a directed set and whenever $F, G \in I$ with $F \leqslant G$, one has $C_G \subseteq C_F$.

Theorem 3.15. *Let S be a dense subsemigroup of $(T, +)$, $e \in E(T)$, and $A \subseteq S$. Let $\mathcal{B} = \{U_n\}_{n=1}^\infty$ has the* **F** *property at the point e. Statements (1), (2), (3), and (4) are equivalent and are implied by statement (5). If S is countable, all five statements are equivalent.*

(1) *A is central near e.*
(2) *There is an FS-tree Tr in A that $\{B_f\colon\ f \in Tr\}$ is collectionwise near e with countable base.*
(3) *There is a $*$-tree Tr in A such that $\{B_f\colon\ f \in Tr\}$ is collectionwise near e with countable base.*
(4) *There is a downward directed family $\langle C_F \rangle_{F \in I}$ of subsets of A such that*
  (i) *for all $F \in I$ and all $x \in C_F$, there exists some $G \in I$ with $C_G \subseteq -x + C_F$, and*
  (ii) *$\{C_F\colon\ F \in I\}$ is collectionwise piecewise syndetic near e with countable base.*
(5) *There is a decreasing sequence $\langle C_n \rangle_{n=1}^\infty$ of subsets of A such that*
  (i) *for all $n \in \mathbb{N}$ and all $x \in C_n$, there is some $m \in \mathbb{N}$ with $C_m \subseteq -x + C_n$, and*
  (ii) *$\{C_n\colon\ n \in \mathbb{N}\}$ is collectionwise piecewise syndetic near e with countable base.*

Proof. See Theorem 4.7 in [8]. □

4. Central sets near e and applications

In this section suppose that T is a commutative semigroup.

Definition 4.1. Let $(T,+)$ be a semitopological semigroup and S be a dense subsemigroup of T. Let $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ has the **F** property at the point $e\in E(T)$.

(a) $\Phi=\{f:\mathbb{N}\to\mathbb{N}:$ for all $n\in\mathbb{N},\ f(n)\leqslant n\}$.
(b) $\mathcal{Y}=\{\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}:$ for each $i\in\mathbb{N}$, $\langle y_{i,t}\rangle_{t=1}^{\infty}$ is a sequence in S and $\sum_{t=1}^{\infty}y_{i,t}$ converges$\}$.
(c) Given $Y=\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ in $\mathcal{Y}$ and $A\subseteq S$, A is a J_Y-set near e if and only if for all $n\in\mathbb{N}$ there exist $a\in U_n$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $\min H\geqslant n$ and for each $i\in\{1,2,\ldots,n\}$, $a+\sum_{t\in H}y_{i,t}\in A$.
(d) Given $Y\in\mathcal{Y}$, $J_Y=\{p\in e_S^*:$ for all $A\in p$, A is a J_Y-set near $e\}$.
(e) $J=\bigcap_{Y\in\mathcal{Y}}J_Y$.

Lemma 4.2. *Let $(T,+)$ be a commutative semitopological semigroup and S be a dense subsemigroup of T. Let $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ has the* **F** *property at the point $e\in E(T)$. Let $Y\in\mathcal{Y}$. Then $K\subseteq J_Y$.*

Proof. Let $Y=\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ and let $p\in K$. To see that $p\in J_Y$, let $A\in p$. To see that A is a J_Y-set near e, let $n\in\mathbb{N}$ be given. For $k\in\mathbb{N}$ let

$$I_k=\Biggl\{\biggl(a+\sum_{t\in H}y_{1,t},a+\sum_{t\in H}y_{2,t},\ldots,a+\sum_{t\in H}y_{n,t}\biggr):\ a\in U_n\cap S,\ H\in\mathcal{P}_f(\mathbb{N}),\ \min H\geqslant k,$$
$$\forall i\in\{1,2,\ldots,n\},\ a+\sum_{t\in H}y_{i,t}\in U_k\Biggr\}$$

and let

$$E_k=I_k\cup\bigl\{(a,a,\ldots,a):\ a\in U_k\cap S\bigr\}.$$

Let $W=\prod_{i=1}^{n}e_S^*$ and let $Z=\prod_{i=1}^{n}\beta S_d$. Let $E=\bigcap_{k=1}^{\infty}cl_Z E_k$ and let $I=\bigcap_{k=1}^{\infty}cl_Z I_k$. Observe that $\emptyset\neq I\subseteq E$. We claim that E is a subsemigroup of W and I is an ideal of E. First to see that $E\subseteq W$, note that for each k, $E_k\subseteq\prod_{i=1}^{n}(U_k\cap S)$ So $cl_{\beta S}E_k\subseteq\prod_{i=1}^{n}cl_Z(U_k\cap S)$ so $E\subseteq W$. Now let $\vec{q},\vec{r}\in E$. We show that $\vec{q}+\vec{r}\in E$ and if either $\vec{q}\in I$ or $\vec{r}\in I$, then $\vec{q}+\vec{r}\in I$. So let $k\in\mathbb{N}$ be given and let U be an open neighborhood of $\vec{q}+\vec{r}$. We show that $U\cap E_k\neq\emptyset$ and if either $\vec{q}\in I$ or $\vec{r}\in I$, then $U\cap I_k\neq\emptyset$. Pick a neighborhood V of $\vec{q}$ such that $V+r\subseteq U$ and pick $\vec{x}\in V\cap E_{2k}$, with $\vec{x}\in I_{2k}$ if $\vec{q}\in I$. If $\vec{x}\notin I_{2k}$, pick $a\in U_{2k}\cap S$ such that $\vec{x}=(a,a,\ldots,a)$ and let $H=\emptyset$. If $\vec{x}\in I_{2k}$, pick $a\in U_{2k}\cap S$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $\min H>2k$, $\vec{x}=(a+\sum_{t\in H}y_{1,t},a+\sum_{t\in H}y_{2,t},\ldots,a+\sum_{t\in H}y_{n,t})$, and each $a+\sum_{t\in H}y_{i,t}\in U_{2k}$. If $H=\emptyset$, let $m=2k$. If $H\neq\emptyset$, let $m=\max H+1$. Now $\vec{x}+\vec{r}\in U$ so pick a neighborhood R of $\vec{r}$ with $\vec{x}+R\subseteq U$. Pick $\vec{w}\in R\cap E_m$ with $\vec{w}\in I_m$ if $\vec{r}\in I$. If $\vec{w}\notin I_m$, pick $b\in U_m\cap S$ such that $\vec{w}=(b,b,\ldots,b)$ and let $G=\emptyset$. If $\vec{w}\in I_m$, pick $b\in U_m\cap S$ and $G\in\mathcal{P}_f(\mathbb{N})$ such that $\min G\geqslant m$, $\vec{w}=(b+\sum_{t\in G}y_{1,t},b+\sum_{t\in G}y_{2,t},\ldots,b+\sum_{t\in G}y_{n,t})$, and each $b+\sum_{t\in G}y_{i,t}\in U_m$. If $G\cup H=\emptyset$, we have $\vec{x}+\vec{w}=(a+b,a+b,\ldots,a+b)\in E_k$. If $G\cup H\neq\emptyset$, then

$$\vec{x}+\vec{w}=\biggl(a+b+\sum_{t\in G\cup H}y_{1,t},a+b+\sum_{t\in G\cup H}y_{2,t},\ldots,a+b+\sum_{t\in G\cup H}y_{n,t}\biggr)\in I_k.$$

Now let $\vec{p}=(p,p,\ldots,p)$. We claim that $\vec{p}\in E$. Let $B\in p$, so that $\prod_{i=1}^{n}cl_{\beta S_d}B$ is a basic neighborhood of $\vec{p}$. Let $k\in\mathbb{N}$ be given. Then $U_k\cap S\in p$ so pick $a\in B\cap U_k$. Then $(a,a,\ldots,a)\in E_k\cap\prod_{i=1}^{n}cl_{\beta S_d}B$. Let M be the smallest ideal of W. By [10, Lemma 2.1], $M=\prod_{i=1}^{n}K$ so $\vec{p}\in M$ and hence $M\cap E\neq\emptyset$. Thus by [1, Corollary 1.2.15] the smallest ideal of E is $E\cap M$ so $\vec{p}$ is in the smallest ideal of E so $\vec{p}\in I$. Thus $I_n\cap\prod_{i=1}^{n}cl_{\beta S_d}A\neq\emptyset$. $\square$

Theorem 4.3. *Let $(T,+)$ be a commutative semitopological semigroup and S be a dense subsemigroup of T. Let $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ has the* **F** *property at the point $e\in E(T)$ and A be a central set near e. Suppose that $Y=\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}\in\mathcal{Y}$. Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in S and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(a) *for each $n\in\mathbb{N}$, $a_n\in U_n$ and $\max H_n<\min H_{n+1}$, and*
(b) *for each $f\in\Phi$, $FS(\langle a_n+\sum_{t\in H_n}y_{f(n),t}\rangle_{n=1}^{\infty})\subseteq A$.*

Proof. See Theorem 4.11 in [8]. $\square$

There are some immediate simple applications of Theorem 4.3. As an example we have the following.

Corollary 4.4. *Let $(T, +)$ be a commutative semitopological semigroup and S be a dense subsemigroup of T. Let $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ has the* **F** *property at the point $e \in E(T)$ and $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S such that $\lim_{n\to\infty} x_n = e$. Assume $r \in \mathbb{N}$ and $S = \bigcup_{i=1}^{r} A_i$. Then there is some $i \in \{1, \ldots, r\}$ such that for every $U \in \tau_e$ and every $l \in \mathbb{N}$ there is an arithmetic progression $\{a, a + d, \ldots, a + ld\} \subseteq A_i \cap U$ with increment $d \in FS(\langle x_n \rangle_{n=1}^{\infty})$.*

Proof. See Corollary 5.1 in [8]. □

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